J. Fluid Mech. (1974), *vol.* 64, *part* 1, *pp.* 195–206
Printed in Great Britain

Turbulent shear flow mixing and rapid chemical reactions: an analogy

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(Received 29 October 1973)

Two-species, irreversible, very rapid reactions, with mild heat release, in a turbulent shear flow are shown to be analogous to the transport of two non-reacting species by the same shear field. Expressions for the probability density functions of the reacting species, the product species and the reaction-generated thermal field are obtained in terms of the joint probability density functions of the two non-reacting species. As an example we have constructed, from recent measurements of temperature statistics at a cross-section in a heated jet, the mean and fluctuating concentration fields of the reacting species and the mean concentration of the product.

1. Introduction

There are a few asymptotic situations in which fluid flows containing chemically reactive species can be successfully studied analytically. Perhaps the most interesting and useful of these is the condition known as 'diffusion limited', by which one means that a time scale characterizing the reaction kinetics is very much less than any appropriate scalar transport time of the flow. Under this condition the reaction proceeds at a rate determined by the fluid transport process rather than chemical kinetics. When there are just two reacting species sharp interfaces develop, separating regions of flow which contain alternately either one or other species but not both. Studies of this phenomenon in laminar flows are fairly extensive, dating back to Danckwerts (1950) and to Pearson (1963). In practice the transporting fluid is usually in turbulent motion and fundamental analytical studies have been confined either to statistically homogeneous flows (O'Brien 1971) or to the local physics and chemistry of the reaction zones (Gibson & Libby 1972; O'Brien 1973). There is also a strong thread of useful research in the chemical engineering literature in which theoretical predictions for chemical mixing in turbulent pipe flow are made on the basis of either numerical modelling or assumptions about the stochastic nature of the concentration fields, e.g. that the fluctuations have a Gaussian distribution. For a review with extensive references the reader is referred to Brodkey (1973).

The purpose of this paper is to pursue an already well-established analogy (Toor 1962) in diffusion-limited turbulent reacting flow to make a direct connexion, without other assumptions, between measurable or potentially measurable quantities in a turbulent mixing shear field and the species concentrations and thermal fields produced by reacting components in the same shear flow.

2. Diffusion-control reactions

We restrict our study to a one-step irreversible reaction of the form

$$A + nB \rightarrow mP + \text{heat} \tag{1}$$

taking place in an incompressible shear flow; n and m are stoichiometric coefficients.

Assuming that the specific heats C_p of all the products are constant and equal and the species molecular diffusivities D are constant and equal and identical to the thermal diffusivity, one obtains the following system of four equations:

$$\left.\begin{aligned} L(\Gamma_A) = W_A, \quad L(\Gamma_B) = nW_A, \quad L(\Gamma_P) = -mW_A, \\ L(\tau) = -W_A \frac{q}{\rho C_p} + \frac{1}{\rho C_p}\frac{DP}{Dt} + \frac{\mu}{\rho C_p}\Phi, \end{aligned}\right\} \tag{2}$$

where Γ_A, Γ_B and Γ_P are the mole concentrations of species A, B and P, respectively, L is the operator $\{\partial/\partial t + \mathbf{u} . \nabla - D\nabla^2\}$, W_A is the rate of production of species A in moles/unit volume per second, ρ is the density, μ the viscosity, τ the temperature, Φ the dissipation function and q the heat released per mole of A produced.

The following approximations are necessary only if the thermal field is to be included in the analogy developed below. Namely, the last two terms of the energy equation must be negligible (Dopazo & O'Brien 1973). That is, neither viscous dissipation nor pressure work contributes significant heat to the system as compared with that produced by the reaction. Both of these approximations become more reasonable at low speeds. It is also necessary to assume that q is a constant: a plausible assumption only if the entire temperature field is not too far from uniform, since in general q depends on the heats of formation of the reacting species and product.

With these approximations and the redefinitions $\Gamma'_B = n^{-1}\Gamma_B$, $\Gamma'_P = \Gamma_P/m$ and $\tau' = \rho C_p \tau/q$ we find, on dropping the primes,

$$L(\Gamma_A) = L(\Gamma_B) = \Gamma(-\Gamma_P) = L(-\tau) = W_A. \tag{3}$$

An alternative formulation, making use of the linearity of the operator L, is as follows (Dopazo & O'Brien 1973; Liñan 1973, private communication):

$$L(\Gamma_B) = W_A, \tag{4a}$$

$$L(\Gamma_B - \Gamma_A) = L(\Gamma_B + \Gamma_P) = L(\Gamma_B + \tau) = 0, \tag{4b–d}$$

where the explicit effect of the reaction is absent from all but the first equation. It is, however, implicit in the convective component of the operator L. In order to remove that implicit dependence it is necessary finally to restrict our consideration to dynamically passive reactive flow systems in which the exchange of energy between the reaction and the flow is negligible. This presumably presupposes mild concentrations or weak heat release and negligible volume change due to reaction. Both of these limits are consistent with the requirements of constant density and small temperature fluctuations already imposed. In deriving (4) no limitation on the rate of reaction has been imposed. However, this formulation leads to a profound simplification only in the diffusion-controlled limit. In this case ($4a$) can be replaced by a simple relation between the probability density function of Γ_B and the probability density function of $\Gamma_B - \Gamma_A$. The method has been described for the special case of homogeneous turbulent flow by O'Brien (1971) and relies essentially on the sharply defined spatial segregation of chemical species when the ratio R of the reaction rate to the rate of turbulent diffusion is very large. Gibson & Libby (1972) and O'Brien (1973) have argued that the interface thickness in such a limit is at least as small as the Batchelor microscale $\eta_B = D/\gamma$, where γ is the strain rate associated with Kolmogorov-scale turbulent eddies (Batchelor 1959).

If we define $\chi(\mathbf{x},t) = \Gamma_B(\mathbf{x},t) - \Gamma_A(\mathbf{x},t)$ then, as a consequence of reactive-species segregation, thin interface surfaces and non-negativeness of concentration,

$$\left.\begin{aligned}
&\lim_{R\to\infty} \overline{\Gamma_A \Gamma_B} = 0,\\
&\lim_{R\to\infty} \Gamma_B(\mathbf{x},t) = \begin{cases} \chi(\mathbf{x},t), & \chi \geqslant 0,\\ 0, & \chi \leqslant 0,\end{cases}\\
&\lim_{R\to\infty} \Gamma_A(\mathbf{x},t) = \begin{cases} -\chi(\mathbf{x},t), & \chi < 0,\\ 0, & \chi > 0.\end{cases}
\end{aligned}\right\} \qquad (5)$$

A more useful consequence, for our purposes, is that

$$\lim_{R\to\infty} P_{\Gamma_B}[\Gamma_B(\mathbf{x},t)] = P_\chi[\Gamma_B(\mathbf{x},t)]\,H(\Gamma_B) + \delta(\Gamma_B)\int_{-\infty}^{0} P_\chi[\chi(\mathbf{x},t)]\,d\chi, \qquad (6)$$

where $H(z)$ is Heaviside's unit function (defined by $H(z) = 1$ for $z > 0$ and $H = 0$ for $z < 0$). The notation $P_y[z]$ signifies the probability density function of the random variable y evaluated at $y = z$.

Equations (4), with (6) replacing ($4a$), give a complete description of species conservation and energy in the diffusion-controlled limit. One property of the system which was expected and which is clearly displayed in this formulation is the lack of any chemical kinetic time scale. Turbulent mixing times, buried in the operator L, determine the evolution of the equations. To solve such a system one must incorporate the Navier–Stokes equation, or its solution, but this is quite beyond us for turbulent shear flows at present. In the next section it is shown that experimental data on the joint probability density function of two simultaneously mixed scalar fields is sufficient to determine the probability density functions of Γ_A, Γ_B, Γ_P and τ.

3. The analogy

Equation (6) provides a method of determining the statistical behaviour of Γ_B in terms of the dependent variable χ of ($4b$) and a similar technique can be applied to determine $P_{\Gamma_A}[\Gamma_A]$. We are left with the task of solving ($4b$–d). It is further evident that if ($4b$) and ($4c$) can be solved for $P_{\Gamma_P}[\Gamma_P]$ the same technique can be used to determine $P_\tau[\tau]$ from ($4d$).

Hence the problem becomes one of constructing an analogy for the pair of equations ($4b$) and ($4c$). Two properties of the operator L are invaluable for this purpose. It is linear and each term contains at least a first-order derivative. Thus one may add an arbitrary constant to each independent variable and multiply it by an arbitrary constant and still satisfy the homogeneous equations (4*b*) and (4*c*).

Consider a boundary surface of a turbulent shear flow. Over the part S_1 of this boundary, species B at a uniform concentration $\Gamma_B(S_1)$ is added. Let species A be injected at a concentration $\Gamma_A(S_2)$ over the part S_2 of the boundary. Across the remainder of the boundary, S_3, the flux of either species is zero. Define

$$Y = \frac{\Gamma_B - \Gamma_A + \Gamma_A(S_2)}{\Gamma_B(S_1) + \Gamma_A(S_2)} = \frac{\chi + \Gamma_A(S_2)}{\Gamma_B(S_1) + \Gamma_A(S_2)}. \tag{7}$$

Then

$$\left.\begin{array}{c} L(Y) = 0, \\ Y = 1 \text{ on } S_1, \quad Y = 0 \text{ on } S_2, \quad \partial Y/\partial \eta = 0 \text{ on } S_3. \end{array}\right\} \tag{8}$$

Consider now an analogous problem in which fluid of temperature $T(S_1)$ is injected over S_1, fluid of temperature $T(S_2)$ is injected over S_2 and the remainder of the boundary, S_3, is thermally insulated. Define

$$\theta = [T - T(S_2)]/[T(S_1) - T(S_2)].$$

Then

$$\left.\begin{array}{c} L(\theta) = 0, \\ \theta = 1 \text{ on } S_1, \quad \theta = 0 \text{ on } S_2, \quad \partial\theta/\partial\eta = 0 \text{ on } S_3. \end{array}\right\} \tag{9}$$

Provided that the velocity field at surfaces of injection of Y is identical to that at surfaces of injection of θ, uniqueness theorems for parabolic systems (Friedman 1963) guarantee that $Y(\mathbf{x}, t) = \theta(\mathbf{x}, t)$ for each member of an ensemble of experiments and hence that

$$P_Y[Y(\mathbf{x}, t)] \equiv P_\theta[\theta(\mathbf{x}, t)]. \tag{10}$$

Using (7) and (5) measurements of $P_\theta[\theta(\mathbf{x}, t)]$ can be converted into $P_{\Gamma_B}[\Gamma_B(\mathbf{x}, t)]$, a task we shall complete at the end of this section.

Defining $z = (\Gamma_B + \Gamma_P)/\Gamma_B(S_1)$, where for convenience we exclude the possibility of injecting the product at any surface although that situation causes no fundamental difficulty, we find

$$\left.\begin{array}{c} L(z) = 0, \\ z = 1 \text{ on } S_1, \quad z = 0 \text{ on } S_2, \quad \partial z/\partial\eta = 0 \text{ on } S_3, \end{array}\right\} \tag{11}$$

and clearly this is analogous to the set (9), so that $P_z[z] = P_\theta[\theta]$. However this is not sufficient information to obtain $P_{\Gamma_P}[\Gamma_P]$ from (5) except where the two random variables Γ_B and Γ_P are statistically independent. In that case the characteristic function of their sum is the product of the characteristic function of Γ_B with that of Γ_P, so that $P_{\Gamma_P}[\Gamma_P]$ can be extracted if $P_{\Gamma_B}[\Gamma_B]$ and $P_z[z]$ are known. There is some suggestion in the literature that this might be a plausible approximation (Dopazo & O'Brien 1973) but there are no data to support it at present.

In order to obtain $P_{\Gamma_P}[\Gamma_P]$ without any assumption it is necessary to define a second turbulent scalar mixing field c, which, for example, might be the concentration of a non-reacting species. Denote by $c(S_1)$ its value when injected across S_1 and, for simplicity, maintain zero concentration on S_2. The flux across the boundary S_3 is zero.

Let $\gamma(\mathbf{x},t) = c(\mathbf{x},t)/c(S_1)$. Then

$$\left.\begin{array}{c} L(\gamma) \equiv 0, \\ \gamma = 1 \text{ on } S_1, \quad \gamma = 0 \text{ on } S_2, \quad \partial\gamma/\partial\eta = 0 \text{ on } S_3, \end{array}\right\} \tag{12}$$

and clearly $P_z[z(\mathbf{x},t)] = P_\gamma[\gamma(\mathbf{x},t)]$, if the velocity field transporting z is identical to that transporting γ. A more useful result for our purposes concerns the relationship of the joint probability density function of the coupled reactive system (Y, Z) to the joint probability density function of the double scalar mixing field (θ,γ). We have

$$P_{YZ}[Y(\mathbf{x},t), Z(\mathbf{x},t)] = P_{\theta\gamma}[\theta(\mathbf{x},t), \gamma(\mathbf{x},t)]. \tag{13}$$

The boundary conditions prescribed in this section are all deterministic. It seems likely from a functional formulation of these problems that (13) also holds where the boundary conditions are stochastic provided that the joint probability density functionals are properly analogous at the boundaries, but we have been unable to find uniqueness theorems which guarantee this result. There is little hope that experiments can be performed at this level of information.

Equation (10) carries information already contained in (13), from which we now derive explicit stochastic information about Γ_A, Γ_B and Γ_P.

From (13), since $Y = [\chi + \Gamma_A(S_2)]/[\Gamma_A(S_2) + \Gamma_B(S_1)]$ and $Z = c/c(S_1)$

$$\left.\begin{array}{l} P_{\chi Z}[\chi, Z] = \dfrac{dY}{d\chi} P_{\theta\gamma}\left[\dfrac{\chi + \Gamma_A(S_2)}{\Gamma_A(S_2) + \Gamma_B(S_1)}, Z\right], \\ \text{or} \\ P_{xZ}[\chi, Z] = \dfrac{c(S_1)\,[T(S_1) - T(S_2)]}{\Gamma_A(S_2) + \Gamma_B(S_1)} P_{Tc}\left[\dfrac{[T(S_1) - T(S_2)]\,[\chi + \Gamma_A(S_2)]}{\Gamma_A(S_2) + \Gamma_B(S_1)} + T(S_2), c(S_1)\,Z\right], \end{array}\right\} \tag{14}$$

where P_{Tc} is the directly measured joint probability density function of the temperature and non-reacting concentration fields.

To obtain $P_{\Gamma_B}(\Gamma_B)$ we first construct $P_\chi[\chi]$ using

$$P_\chi[\chi] = \int_{-\infty}^{+\infty} P_{\chi Z}[\chi, Z]\, dZ.$$

Therefore, from (14),

$$P_\chi[\chi] = \frac{T(S_1) - T(S_2)}{\Gamma_A(S_2) + \Gamma_B(S_1)} P_T\left[\frac{[T(S_1) - T(S_2)]\,[\chi + \Gamma_A(S_2)]}{\Gamma_A(S_2) + \Gamma_B(S_1)} + T(S_2)\right].$$

Finally, on applying (5) we find

$$\begin{aligned} P_{\Gamma_B}[\Gamma_B] = {} & \frac{T(S_1) - T(S_2)}{\Gamma_A(S_2) + \Gamma_B(S_1)} P_T\left[\frac{[T(S_1) - T(S_2)]\,[\Gamma_B + \Gamma_A(S_2)]}{\Gamma_A(S_2) + \Gamma_B(S_1)} + T(S_2)\right] H(\Gamma_B) \\ & + \delta(\Gamma_B) \int_{-\infty}^{0} P_T\left[\frac{[T(S_1) - T(S_2)]\,[\chi + \Gamma_A(S_2)]}{\Gamma_A(S_2) + \Gamma_B(S_1)} + T(S_2)\right] d\chi, \end{aligned} \tag{15}$$

where $P_T[T(\mathbf{x},t)]$ is a directly measured probability density function of the temperature. A similar result is easily generated for $P_{\Gamma_A}[\Gamma_A]$.

To obtain an expression for P_{Γ_P} we construct the joint probability density function $P_{\Gamma_B\Gamma_P}[\Gamma_B, \Gamma_P]$ with the help of (5):

$$P_{\Gamma_B\Gamma_P}[\Gamma_B, \Gamma_P] = \frac{1}{\Gamma(S_1)} P_{\chi Z}\left[\chi, \frac{\chi+\Gamma_P}{\Gamma_B(S_1)}\right] H(\chi) + \frac{\delta(\chi)}{\Gamma_B(S_1)} \int_{-\infty}^{0} P_{\chi Z}\left[\chi, \frac{\Gamma_P}{\Gamma_B(S_1)}\right] d\chi.$$

Hence, with (14),

$$\begin{aligned} P_{\Gamma_B\Gamma_P}[\Gamma_B, \Gamma_P] = {} & \frac{T(S_1)-T(S_2)}{\Gamma_A(S_2)+\Gamma_B(S_1)} \frac{c(S_1)}{\Gamma_B(S_1)} \left\{ P_{Tc}\left[\frac{[T(S_1)-T(S_2)][\Gamma_B+\Gamma_A(S_2)]}{\Gamma_A(S_2)+\Gamma_B(S_1)} \right.\right. \\ & \left. + T(S_2), \frac{c(S_1)(\Gamma_B+\Gamma_D)}{\Gamma_B(S_1)} \right] H(\Gamma_B) + \delta(\Gamma_B) \int_{-\infty}^{0} P_{Tc}\left[\frac{[T(S_1)-T(S_2)][\chi+\Gamma_A(S_2)]}{\Gamma_A(S_2)+\Gamma_B(S_1)} \right. \\ & \left.\left. + T(S_2), \frac{c(S_1)\Gamma_P}{\Gamma_B(S_1)} \right] \right\}. \end{aligned} \tag{16}$$

By integration of (16) with respect to Γ_P one recovers (15). Similarly, to obtain $P_{\Gamma_P}[\Gamma_P]$ it is necessary to integrate (16) with respect to Γ_B. Since Γ_B occurs in both independent variables no simplification in the formalism results and in general a numerical integration of the experimental data on the joint probability density function $P_{Tc}[T, c]$ is necessary. The mean concentration of the product $\overline{\Gamma}_P[\mathbf{x}, t)$ is an exception. It may be directly obtained from the relationship $Z = \gamma$. That is,

$$(\overline{\Gamma}_B+\overline{\Gamma}_P)/\Gamma_B(S_1) = \overline{c}/c(S_1),$$

or

$$\overline{\Gamma}_P(\mathbf{x}, t) = \Gamma_B(S_1)\,\overline{c}(\mathbf{x}, t)/c(S_1) - \overline{\Gamma}_B(\mathbf{x}, t).$$

Entirely analogous results apply for determining the statistical properties of the reaction-produced temperature field τ with one added proviso. If the reactant species are introduced at the same constant temperature τ_0, then we find that $P_{\Gamma_B\tau}[\Gamma_B, \tau]$ can be obtained from $P_{\Gamma_B\Gamma_P}[\Gamma_B, \Gamma]$ by a simple substitution of $\tau-\tau_0$ for the independent variable Γ_p:

$$P_{\Gamma_B\tau}[\Gamma_B, \tau-\tau_0] = P_{\Gamma_B\Gamma_P}[\Gamma_B, \tau-\tau_0].$$

4. Some examples

In this section we describe five typical turbulent shear flows, where the above analogy is applicable. The examples chosen are by no means exhaustive; they are meant to suggest the possible usefulness of the analogy and in fact they represent systems that have been used in previous reactive-flow investigations.

4.1. *The two- or three-dimensional wake*

A uniform stream with a reactant concentration of uniform value $\Gamma_A(S_2)$ flows over a sphere from which a reactant species of concentration $\Gamma_B(S_1)$ is injected into the main stream. The temperatures at S_1, S_2 and in the uniform stream are τ_0.

The analogy consists of a uniform stream at constant temperature $T(S_2)$ flowing over the sphere from which is injected a non-reactant species of concentration $c(S_1)$ at temperature $T(S_1)$. The surface of the sphere is thermally insulated.

4.2. *The two- or three-dimensional jet*

A reactant B at concentration $\Gamma_B(S_1)$ is injected into a main flow containing a species A at a uniform concentration $\Gamma_A(S_2)$. Both temperatures are τ_0.

The analogous system consists of a fluid of concentration $c(S_1)$ at temperature $T(S_1)$ injected into a main flow of zero concentration and temperature $T(S_2)$.

4.3. *The two-dimensional mixing layer in the wake of a flat plate*

The upper stream contains species A at a concentration $\Gamma_A(S_2)$ and the lower stream contains species B at a concentration $\Gamma_B(S_1)$. Both temperatures are τ_0.

An analogy would consist of an upper stream at zero concentration and a temperature $T(S_2)$ whereas the lower stream would have a temperature $T(S_1)$ and concentration $c(S_1)$.

4.4. *Two coaxial jets*

Species B is injected from the inner jet with concentration $\Gamma_B(S_1)$. Species A is injected from the outer jet with concentration $\Gamma_A(S_2)$. The temperature of both streams and the environment is τ_0.

The analogous system is comprised of an inner jet with concentration $c(S_1)$ and temperature $T(S_1)$. The outer jet has zero concentration and temperature $T(S_2)$. The environment temperature, say T_∞, in this case can be shown to have to satisfy the following constraint:

$$T_\infty\left[1+\frac{\Gamma_A(S_2)}{\Gamma_B(S_1)}\right] = T(S_2)+\frac{\Gamma_A(S_2)}{\Gamma_B(S_1)}\,T(S_1). \tag{17}$$

4.5. *The sphere wake with two-species injection*

A uniform stream of temperature τ_0 flows over a sphere. Species B is injected from a region S_1 of the sphere at temperature τ_0 and concentration $\Gamma_B(S_1)$. Species A is injected from a region S_2 of the sphere with concentration $\Gamma_A(S_2)$ and the same temperature τ_0.

The analogue consists of a uniform stream at temperature T_∞ and zero concentration flowing over the same sphere, from which a non-reactant species is injected at a concentration $c(S_1)$ and temperature $T(S_1)$ from a region S_1 of the surface. From a region S_2 of the surface a fluid with zero concentration and temperature $T(S_2)$ is emitted. The constraint (17) again applies and the sphere is thermally insulated.

5. An application

There are several recent experimental investigations which have generated data on probability density functions of temperature in shear flows but to our knowledge there are no data on the joint probability density function of two scalar fields. In this section we apply the analogy to the case given in §4.2, making use of recently reported measurements of the temperature across a heated axisymmetric jet, with exit Reynolds number $31{\cdot}6\times10^4$ (Tutu & Chevray 1973).

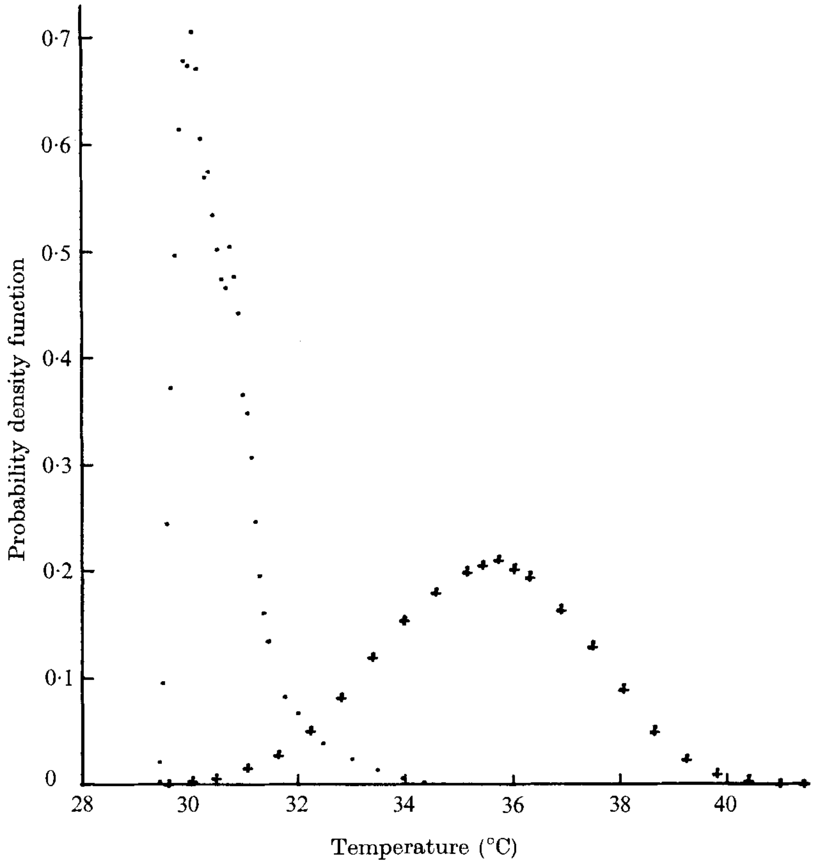

FIGURE 1. Probability density functions of temperature 15 diameters downstream of jet exit. +, 1 diameter from centre-line; ●, $\frac{8}{3}$ diameters from centre-line.

The exit temperature $T(S_1)$ of the jet was 50·4 °C, the environmental temperature was 29·26 °C and the jet diameter was 9 in. Measurements were taken at eight radial positions 15 diameters downstream of the jet. The analogy described in §3 can be used to transform measured $P_\theta[\theta]$ into $P_{\Gamma_B}[\Gamma_B]$ and $P_{\Gamma_A}[\Gamma_A]$. Further, the mean concentration of product (or reaction-generated temperature) can be found from these measurements, as we showed in §3. The relationship is

$$\overline{\Gamma}_P = \Gamma_B(S_1)\,[\overline{T} - T(S_2)]/[T(S_1) - T(S_2)] - \overline{\Gamma}_B.$$

More details on product-species behaviour can only be found by measurement of joint probability density functions.

The results are presented in terms of the mean and fluctuating fields of Γ_A and Γ_B, and the mean of Γ_P as a function of radial position at the 15 diameter axial location. Figure 1 shows the measured temperature probability density function at two radial locations. In figure 2 the mean fields are displayed for various values of $\Gamma_B(S_1)$ when $\Gamma_A(S_2) \equiv 1$. When $\Gamma_B(S_1) = \Gamma_A(S_2)$ there is almost no

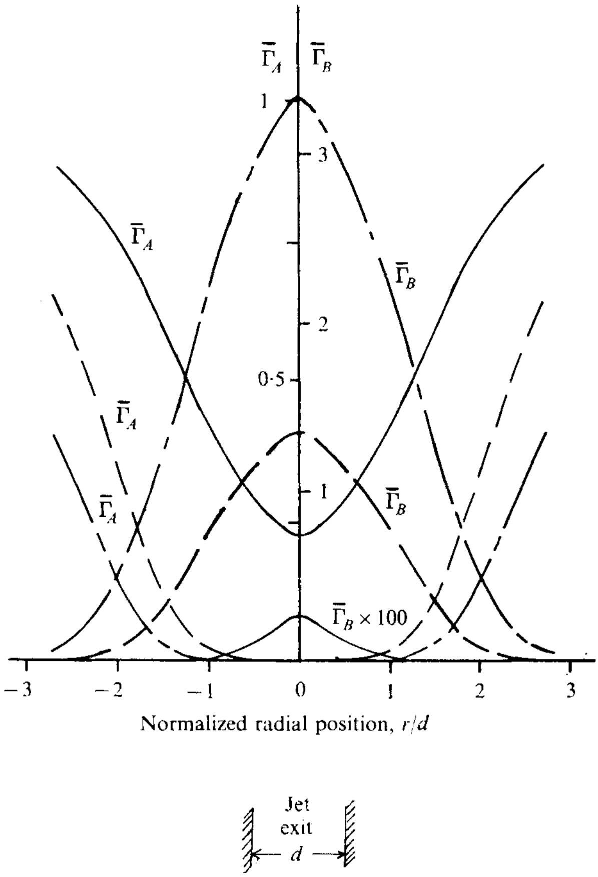

FIGURE 2. Mean concentration of reacting species 15 diameters downstream of jet exit as a function of radial position. $\Gamma_A(S_2) = 1$. ——, $\Gamma_B(S_1) = 1$; — —, $\Gamma_B(S_1) = 5$; – – –, $\Gamma_B(S_1) = 10$.

concentration of B at 15 diameters downstream and that curve has been amplified 100 times in order to display it on the same graph as

$$\Gamma_B(S_1) = 5 \quad \text{and} \quad \Gamma_B(S_1) = 10.$$

The same amplification has been applied to the fluctuation intensity $\overline{\gamma_B^2}$ in figure 3. The mean concentration of product is shown in figure 4.

It is clear from figure 1 that there is no simple similarity assumption by which one can reasonably give an analytical form to the temperature probability density function. This will be even more true closer to the jet exit except possibly in the mixing-layer region immediately after the jet. There seems to be no alternative to extensive measurements of probability density functions and joint probability density functions in the entire thermal field of the heated jet. But such measurements can be extended, by this analogy, to describe the statistics of the concentration fields of rapidly reacting species, a task which is generally considered to be very difficult experimentally.

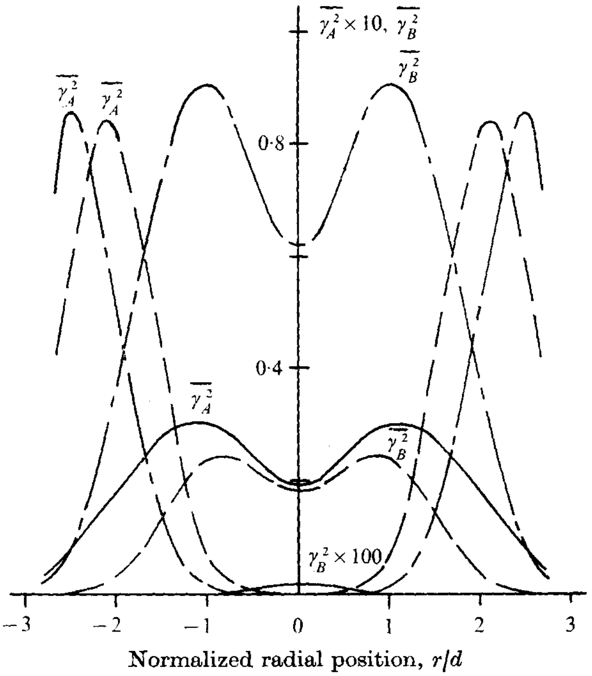

FIGURE 3. Intensity of concentration fluctuations of reacting species 15 diameters downstream of jet exit as a function of radial position. $\Gamma_A(S_2) = 1$. ——, $\Gamma_B(S_1) = 1$; — —, $\Gamma_B(S_1) = 5$; — — —, $\Gamma_B(S_1) = 10$.

6. Discussion

The analogy developed above is, in principle, exact. However, when it is used with experimental data as in §5 the unavoidable imprecisions in the boundary conditions of one system may not behave analogously in the other. For example, the temperature of the environment in the jet experiment referred to in §5 is not deterministic but has a fluctuation level of the order of 1 °C (Tutu & Chevray 1973). On the other hand the chemical-species concentration Γ_B must be strictly zero in the ambient portions of the chemically reacting jet. Thus only negative fluctuations of the random variable χ with respect to $\Gamma_A(S_2)$ are permissible. We have, for the purposes of the analogue, interpreted the minimum measured temperature (29·26 °C) as the ambient one. Fluctuations with respect to that value are positive everywhere. Hence no arbitrariness arises in carrying out the computations of the analogue but some inaccuracy, especially near the chemical jet boundary, must be anticipated. Our plotted results avoid the boundary. An alternative method of processing the data would have been to discard all temperatures whose values fall below the mean ambient temperature and to normalize the probability density functions shown in figure 1. All temperatures so discarded could be identified with the fluctuations of Γ_A in the ambient fluid. This procedure is more complicated and not obviously more accurate than the one we have adopted. Experiments exhibiting the lowest possible fluctuations

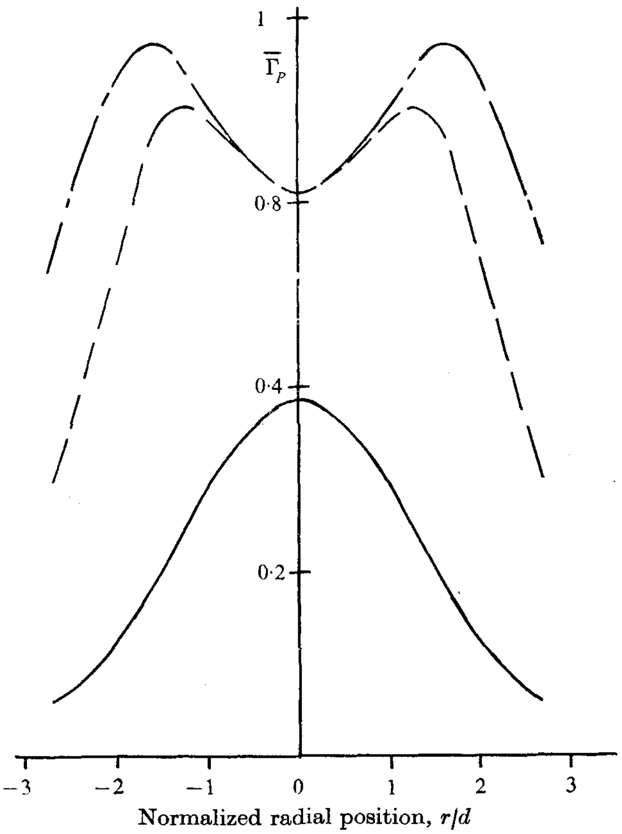

FIGURE 4. Mean concentration of product species 15 diameters downstream of jet exit as a function of radial position. $\Gamma_A(S_2) = 1$. ——, $\Gamma_B(S_1) = 1$; — —, $\Gamma_B(S_1) = 5$; – – –, $\Gamma_B(S_1) = 10$.

in ambient temperature would be desirable in any further application of the theory. Recent results (La Rue & Libby 1974) with a two-dimensional heated wake are excellent in this respect and we have applied our technique to them. Unfortunately the measured data points are too sparse (eight to a cross-section) and the initial temperature differences too large for our method to be reliable. If the reader should care to see the results of this application he may obtain them by writing to either of the authors of this paper.

We wish to thank Prof. Chevray and Prof. Libby for allowing us to make use of unpublished data from their laboratories. The National Science Foundation supported this research under grant GK-21214.

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